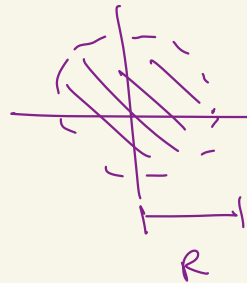


Friday, Oct 3, Lecture 15: Powers, exp, log

Last time... complex functions $f: \mathbb{C} \rightarrow \mathbb{C}$.

Power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ on radius of convergence}$$



$$e^{iz} = \cos z + i \sin z$$

$$\Rightarrow \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\text{when } z = iy \in i\mathbb{R},$$

$$\sin iy = \frac{e^{-y} - e^y}{2i} =: i \sinh(y)$$

$$\cos iy = \frac{e^{-y} + e^y}{2} =: \cosh(y)$$

Thm (De Moivre)

Let $z = re^{i\theta}$. Then

$$z^n = (re^{i\theta})^n = \underbrace{r^n}_{\substack{\text{rescale} \\ \text{by } |z| \mapsto |z|^n}} \underbrace{e^{in\theta}}_{\substack{\text{rotate by } n\theta \\ \text{angle.}}}$$

$$= r^n (\cos n\theta + i \sin n\theta)$$

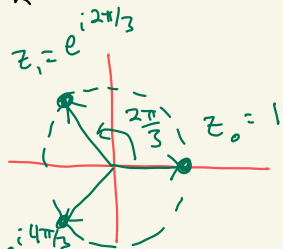
cor: (roots). Assume $w \neq 0$,

There are exactly n solutions to $z^n = w = re^{i\theta}$

$$z_k = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}, \quad k=0, 1, 2, \dots, n-1$$

Ex: $z^3 = 1$

$\Rightarrow |z| = 1$
 $\arg z = 0$



$$z_0 = 1 \cdot e^{i(0+0)} = 1$$

$$z_1 = 1 \cdot e^{i(0 + \frac{2\pi}{3})} = e^{i\frac{2\pi}{3}}$$

$$z_2 = e^{i\frac{4\pi}{3}}$$

check:

$$z_k^n = r^{n/n} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n}) \cdot n} = re^{i(\theta + 2\pi k)} = re^{i\theta} \text{ by periodicity.}$$

• After $k=n-1$, it starts repeating itself:

$$z_n = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2\pi n}{n})} = r^{1/n} e^{i\frac{\theta}{n}} e^{i2\pi} = r^{1/n} e^{i\frac{\theta}{n}} = z_0$$

and likewise $z_m = z_{m \bmod n}$, and there are only n integers mod n .

• There are no other solutions b-y factoring:

$$(z^n - w) = (z - z_0)(z - z_1) \dots (z - z_{n-1})$$

$\nwarrow$ degree n $\nwarrow$ degree n

Prop: Properties of e . Let $x, y \in \mathbb{R}$ and $z, w \in \mathbb{C}$

i) $e^{z+w} = e^z e^w$

ii) $|e^{x+iy}| = e^x |e^{iy}| = e^x$

iii) $\arg(e^{x+iy}) = y \pmod{2\pi}$

iv) $e^z \neq 0$ for all $z \in \mathbb{C}$.

v) $e^z = 1$ iff $z = 2\pi i n$ for some integer n .

vi) e^z is periodic with period $2\pi i$, i.e.,

$$e^{z+2\pi i n} = e^z \quad \text{for all } n \in \mathbb{Z}$$

Gives a new polar form:

$$z = \underbrace{r}_{e^x} \underbrace{e^{i\theta}}_{e^{iy}} = r(\cos\theta + i\sin\theta) \quad z \neq 0.$$

Prop:

$$\frac{d}{dz} e^z = e^z$$

← This is a complex derivative:

$$\left(\frac{d}{dz} f\right)(w) := \lim_{h \rightarrow 0} \frac{f(w+h) - f(w)}{h}$$

will say more later!

Pf:

For polynomials,

$$\frac{d}{dz} z^n = n z^{n-1}$$

$$\begin{aligned} \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} &= \frac{d}{dz} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) \\ &= 0 + 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \\ &= e^z \end{aligned}$$



We want to define logarithm. It should be the inverse of e :

$$e^{\log z} = z$$

$$\log e^z = z.$$

Remember,

$$e^{x+iy} = e^x e^{iy}$$

$\uparrow$ $\uparrow$
 r $e^{i\theta}$

So guess:

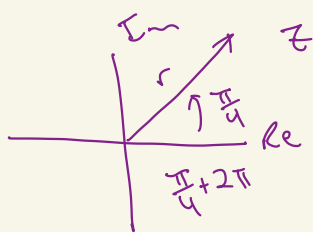
$$\begin{aligned} \log(z) &= \log(e^x e^{iy}) = \log(e^x) + \log(e^{iy}) \\ &= \log|z| + i \arg(e^{iy}) \end{aligned}$$

$\uparrow$ $\uparrow$
 $\log \text{ length}$ $\log \text{ angle.}$

$\arg(e^{iy}) = iy.$

But there's a catch: we have to choose a branch of argument, meaning $\arg(z) \in [a, a+2\pi)$

this choice of interval is the branch



$$z = r e^{i\theta} = r e^{i(\theta+2\pi)}$$

$$\Rightarrow \log z = \log r + i \frac{\pi}{4}$$

[0, 2\pi)

$$\log z = \log r + i \left(\frac{\pi}{4} + 2\pi \right)$$

[2\pi, 4\pi)

This is why historically people called $\log$ a "multivalued function". But we like functions, so we generally pick a specific branch like $[-\pi, \pi)$ or $[0, 2\pi)$ to get a function.

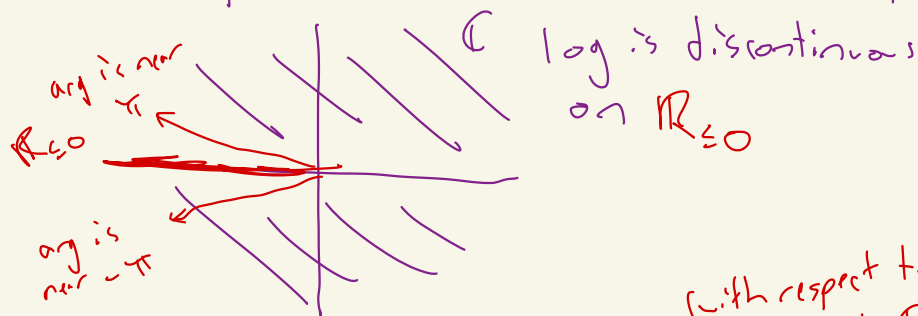
- This is a familiar idea from trig:

$\tan \theta = y$ has many solutions,

$\theta + 2\pi n$. So we demand

$\arctan: \mathbb{R} \rightarrow (-\pi/2, \pi/2)$ so $\theta \in (-\pi/2, \pi/2)$.

- Note that argument (and so $\log$) is discontinuous at the endpoint! So if we choose $[-\pi, \pi)$,



(with respect to the branch cut $\mathbb{R}_{\le 0}$)

Def:

The principal branch of the logarithm is

$\log: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R} \times i[-\pi, \pi)$ defined by

$$\log z = \log |z| + i \arg z, \quad \arg z \in [-\pi, \pi).$$

(all other branches differ by $\log z = \widetilde{\log} z + 2\pi i k, k \in \mathbb{Z}$.)

Bourbaki prefers $[0, 2\pi)$.

Ex:

What are the possible solns to

$$e^z = 1$$

Soln:

By periodicity,

$$z = i2\pi k, \quad k \in \mathbb{Z}$$

$$= \dots, -2\pi i, 0, 2\pi i, \dots$$

• choose the principal branch $[0, 2\pi)$ of $\log$.

Then

$$\log(1) = \log|1| + i\arg(1)$$

$$= 0$$

• choose the branch $[2\pi, 4\pi)$. Then

$$\tilde{\log}(1) = \log|1| + i\tilde{\arg}(1)$$

$$= 0 + i2\pi$$